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MPO 663 – Convective and Mesoscale Met.
Homework 2

The point of this problem set is to illustrate the *Rossby radius of deformation*, and its dependence on vertical structure. We use the simplest possible framework for a heating-induced balanced vortex: a hydrostatic Boussinesq stratified atmosphere, with perturbations linearized about a state of rest (no advection terms). Thus, ρ = constant (except that there is a buoyancy b), so that the pressure gradient force is written as $\phi_x \equiv p'_x/\rho$. (I used π instead of ϕ in class.)

Please **understand** our equation set:

$u_t = fv - \phi_x$	u momentum equation
$v_t = -fu - \phi_y$	v momentum equation
$0 = b - \phi_z$	w momentum equation (hydrostatic)
$u_x + v_y + w_z = 0$	continuity equation
$b_t = -N^2 w + q$	thermodynamic equation w/ heating ($q = g/\theta \, d\theta/dt$)

We study solutions with separable vertical and horizontal structure. For this step, it is simplest to go to 2 dimensions (ignore y variations) and to set aside the Coriolis term, which can be trivially added back in later. This simplifies the equation set to 4. **Understand** them:

$u_t = -\phi_x$	u momentum equation
$0 = b - \phi_z$	w momentum equation (hydrostatic)
$u_x + w_z = 0$	continuity equation
$b_t = -N^2 w + q$	thermodynamic equation w/ heating q

1. Assume solutions of the form $u = U(x,t)\cos(mz)$, $\phi = \Phi(x,t)\cos(mz)$, $w = W(x,t)\sin(mz)$, $b = B(x,t)\sin(mz)$, $q = Q(x,t)\sin(mz)$.

a) (i) **Confirm** that these satisfy the boundary condition w=0 at z=0 for all vertical wavenumbers m.
At z=0, w=W(x,t)sin(m(0))=W(x,t)[0]=0

(ii) **Plug** these forms into the 4 equations above,
$u_t = -\phi_x \leftrightarrow U_t(x,t) = -\Phi_x(x,t)$
$0 = b - \phi_z \leftrightarrow 0 = B(x,t) + m\Phi(x,t)$
$u_x + w_z = 0 \leftrightarrow U_x(x,t) + mW(x,t) = 0$
$b_t = -N^2 w + q \leftrightarrow B_t(x,t) = -N^2 W(x,t) + Q(x,t)$
(iii) then **substitute** the hydrostatic equation into the thermo equation
$-m\Phi_t(x,t) = -N^2 W(x,t) + Q(x,t)$
(iiii) and the result into the continuity equation to **show:**

$U_t = -\Phi_x$ u momentum equation
$\Phi_t = -N^2/m^2\,(U_x + \delta_d)$ height equation w/ source

where the heating is expressed as *diabatic divergence* $\delta_d \equiv \partial/\partial z(q/N^2) = -mQ/N^2$.
$U_t(x,t) = -\Phi_x(x,t)$ from (i) above
$W(x,t) = \frac{1}{N^2}\left(m\Phi_t(x,t) + Q(x,t)\right)$
$U_x + mW = U_x(x,t) + \frac{m}{N^2}\left(m\Phi_t(x,t) + Q(x,t)\right) = 0$
$\Phi_t(x,t) = \frac{N^2}{m^2}\left(-U_x(x,t) + \frac{m}{N^2}\,Q(x,t)\right) = -\frac{N^2}{m^2}\left(U_x(x,t) - \frac{mQ(x,t)}{N}\right)$
$\Phi_t(x,t) = -\frac{N^2}{m^2}\left(U_x(x,t) + \delta_d(x,t)\right)$

b) (i) **Rewrite** these to make clear that they are the equations for forced waves with free-surface height amplitude $h = \Phi/g$, in a shallow (compared to wavelength) layer of water (or any other inviscid fluid), with *equivalent depth* $H = N^2/(gm^2)$.

Plugging in Φ=hg and $gH = N^2/m^2$ into the two highlighted equation above gives:
$U_t = -(gh)_x = -gh_x$ (u momentum equation)
$\Phi_t = -gH(U_x + \delta_d)$ (height equation w/ source)

(ii) **Confirm** that the v equation, Coriolis terms, and u and v momentum source terms could have been carried along in the same way to give the forced shallow water set for a given value of m:
$\Phi_t = -gH(U_x + \delta_d)$
$U_t = fV - \Phi_x + F$ u momentum equation

V_t = -fU -Φ_y + G v momentum equation

Φ_t = -N^2/m^2 (U_x + V_y + δ_d) how divergence changes Φ

(i)

First, Let u = U(x,y,t) cos(mz), v = V(x,y,t) cos(mz),

ϕ = Φ(x,y,t) cos(mz), w = W(x,y,t) sin(mz),

b = B(x,y,t) sin(mz), q = Q(x,y,t) sin(mz)

At z=0, w=W(x,y,t)sin(m(0))=W(x,y,,t)[0]=0

(ii)

u_t = fv - ϕ_x + X $\leftrightarrow$ $U_t(x,y,t) = fV(x,y,t) - \Phi_x(x,y,t) + F$

v_t = -fu - ϕ_y+Y $\leftrightarrow$ $V_t(x,y,t) = -fU(x,y,t) - \Phi_x(x,y,t) + G$

0 = b - ϕ_z $\leftrightarrow$ $0 = B(x,y,t) + m\Phi(x,y,t)$

u_x + v_y + w_z = 0 $\leftrightarrow$ $U_x(x,y,t) + V_y(x,y,t) + mW(x,y,t) = 0$

b_t = -N^2w + q $\leftrightarrow$ $B_t(x,y,t) = -N^2W(x,y,t) + Q(x,y,t\,)$

Note: X, Y, F, and G represent friction.

(iii)

$-m\Phi_{\mathrm{t}}(x,y,t) = -N^2W(x,y,t) + Q(x,y,t\,)$

(iiii)

$W(x,y,t) = \frac{1}{N^2}(Q(x,y,t\,) + m\Phi_{\mathrm{t}}(x,y,t))$

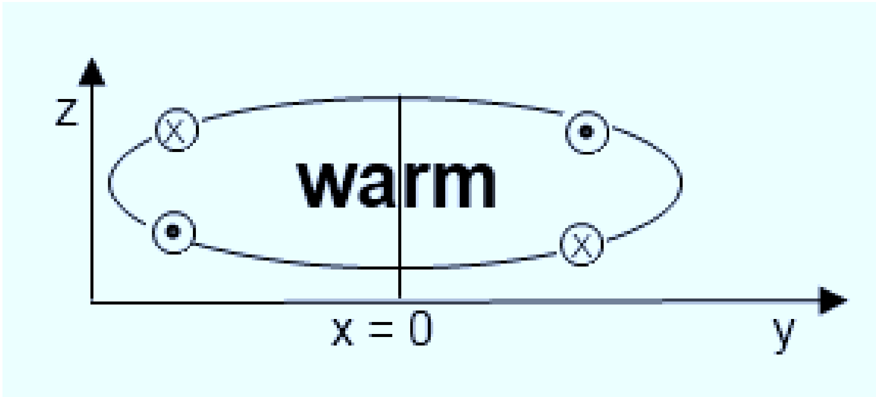
U_x + V_y + mW $= U_x(x,y,t) + V_y(x,y,t) + \frac{m}{N^2}\left(m\Phi_{\mathrm{t}}(\mathrm{x,y,t}) + Q(x,y,t\,)\right)$

$= 0;\;\; \Phi_{\mathrm{t}}(\mathrm{x,y,t}) = \frac{N^2}{m^2}\left(-U_x(x,y,t) - V_y(x,y,t) + \frac{m}{N^2}\,Q(y,x,t\,)\right)$

$= -\frac{N^2}{m^2}\left(U_x(x,y,t) + V_y(x,y,t) - \frac{mQ(x,y,t\,)}{N}\right)$

$\Phi_{\mathrm{t}}(\mathrm{x,y,t}) = -\frac{N^2}{m^2}\left(U_x(x,y,t) + V_y(x,y,t) + \delta_d(x,y,t)\right)$

2. Now let’s solve for the structure of an inviscid (F=G=0) balanced line vortex (shear line) created by a *past* heating along x = 0 on an f-plane, with f = 10^{-4} s^{-1}. Suppose that the heating which created the vortex was a line of heating along x=0, and only excited a single vertical wavenumber m. Sketch this flow in the x-z plane for m=1/2 and z domain [0,1]. (It is like the cross section of a hurricane: a warm core vortex).



a) Confirm that this balanced flow should have no y or t derivatives (i.e., the flow is steady state) with $\delta_d = 0$ at the current time.

-Since the heat source at x = 0 was infinitely long in the y-direction, there is no y-derivative in the current flow.

-Since the heating occurred at some time in the past, all the gravity waves are already gone and the flow is steady-state in time, and thus there is no t-derivative. Also, since the heating occured in the past, $\delta_d = -mQ/N^2 = 0$ at the current time.

b) (i) **Write** the steady state version of the 3-equation set above.

$U_t(x,y,t) = fV(x,y,t) - \Phi_x(x,y,t) + F \rightarrow \Phi_x(x) = fv(x)$

$V_t(x,t) = -fU(x,t) - \Phi_x(x,t) + G \rightarrow \Phi_x(x) = -fu(x)$

$\Phi_t(x,y,t) = -\frac{N^2}{m^2}\left(U_x(x,y,t) + V_y(x,y,t) + \delta_d(x,y,t)\right) \rightarrow u_x(x) = 0$

(ii) **What, in words, remains of the equation set's physical meaning?**

Geostrophic balance

(iii) Does it give us the structure of the vortex?

No.

(iiii) Does it contain the information that we specifically seek the vortex *created by a past heating along x = 0*? Why not?

No. While the geostrophic balance obtained above is valid, this result is ambiguous, as the content/information of the thermodynamic equation with heating is lost when the y and t derivatives are cancelled, plus the height equation, $\Phi_t = -gH(U_x + V_y + \delta_d)$, becomes just 0 = 0.

3. To proceed, we need a different tactic. **Derive** the potential vorticity conservation equation for the 2D shallow water set from 1b (with F=G=Q=0), by **constructing** a vorticity equation from U and V and **substituting** divergence from the height equation.

$u_t = fv - \phi_x$, $v_t = -fu - \phi_y$, $u = U(x,y,t)\cos(mz)$, $v = V(x,y,t)\cos(mz)$,

$\phi = \Phi(x,y,t)\cos(mz)$, $\zeta = v_x - u_y$

$U_t = fV - \Phi_x, V_t = -fU - \Phi_y$

$\zeta_t = \left(V_x - U_y\right)_t = (V_t)_x - (U_t)_x = \left(-fU_x - \Phi_{yx}\right) - \left(fV_y - \Phi_{xy}\right)$

$\zeta_t = -f\left(U_x + V_y\right)$ vorticity equation

$h = \Phi/g$; $\Phi = gh$; $\Phi_t = -gH\left(U_x + V_y + \delta_d\right)$; $\delta_d = 0$

$gh_t = -gH\left(U_x + U_y\right)$; $h_t = -H\left(U_x + U_y\right)$ height equation

$\frac{\zeta_t}{f} = -\left(U_x + V_y\right)$; $\frac{\zeta_t}{f} = \frac{h_t}{H}$ PV equation

4.

a) To define the balanced vortex structure, **express mathematically** the statement that PV is constant and equal to its pre-heating value everywhere except along the line x=0 where the heating occurred.

$\frac{\zeta_t}{f} = \frac{h_t}{H} = \frac{f}{H}$; $\zeta_t = \frac{f^2}{H} = \frac{fh_t}{H} = constant\ in\ t \rightarrow h_t = constant\ in\ t$

$\rightarrow \frac{\zeta}{f} = \frac{h_t}{H}(t) = \frac{h}{H}$ (PV equation, time integrated)

$V = V_g = \frac{1}{f}\frac{\partial \Phi}{\partial x} = \frac{g}{f}h_x$ (V is geostrophic)

$V_x = \frac{g}{f}h_{xx}$; $\frac{\zeta}{f} = \frac{V_x - U_y}{f}$; $U_y = 0$; $\frac{\zeta}{f} = \frac{V_x}{f} = \frac{h}{H}$; $V_x = \frac{g}{f}h_{xx} = \frac{hf}{H}$

$h_{xx} = \frac{f^2}{gH}h$

Guess: $h = h_0 e^{-\frac{f}{\sqrt{gH}}x}$ (picked the decaying solution, so that $h_x < 0$)

Proof: $h_x = -h_0\frac{f}{\sqrt{gH}}e^{-\frac{f}{\sqrt{gH}}x}$; $h_{xx} = \frac{f^2}{gH}\left(h_0 e^{-\frac{f}{\sqrt{gH}}x}\right) = \frac{f^2}{gH}h$ Confirmed!

So, $PV = \frac{\zeta}{f} = \frac{h}{H}$ is constant and equal to its preheating value of $\frac{h_0}{H}$ along x = 0.

b) **Solve** this equation for Φ(x) and V(x). **Write** the horizontal length scale for the vortex's exponential decay structure: this length is called the *Rossby deformation radius*. What is its dependence on m?

$L_R = \frac{\sqrt{gH}}{f}$; Let $c = \sqrt{gH}$ = phase speed, then $L_R = c/f$

$H = \frac{N^2}{gm^2}$; $c = \sqrt{g\left(\frac{N^2}{gm^2}\right)} = \frac{N}{m}$